Approximate Analysis of a Laminar Elliptical Jet

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Theme

THE past several years have witnessed an upsurge of interest in thrust augmenting ejectors for V/STOL aircraft and ejector pumps for lasers. In each of these devices, as well as many others, energy is transferred from a high velocity primary flow to a lower velocity secondary flow in such a compact dimension that the near field of the jet is critical. A parameter of basic importance in determining the success or failure of this transfer of energy is the momentum distribution due to the shape and location of the primary nozzles. The momentum distribution may be characterized by a "half width" (defined as the lateral position of the mean velocity), since a majority of the jet's momentum is contained within the half widths. In the case of turbulent nonaxisymmetric jet shapes, Trentacoste and Sforza[1] have shown experimentally that the major axis half width initially decreases while on the minor axis, the half width grows monotonically. Such a difference in growth rate has not been predicted by either a laminar or turbulent treatment of the problem. Pai and Hsieh[2] have treated the linearized case of the laminar rectangular jet by applying the boundary-layer assumptions in both the major and minor axis directions. The result is a monotonic half width growth on both axes.

In the present analysis, the Navier-Stokes equations are derived in elliptical coordinates. It is then assumed that at any position within the jet, one of the orthogonal coordinate surfaces coincides with the elliptical isovel passing through that position. At this point it is possible to make some rational assumptions concerning the functional form and order of magnitude of each of the variables employed. These assumptions reduce the equations to boundary-layer (i.e., parabolic) type. They also reduce the 4 equations and 4 unknowns to 2 equations and 2 unknowns. These 2 equations are of two-dimensional form with several additional terms, and may be described as quasi-two-dimensional. This form does not require that the equations be solved throughout the three-dimensional field but rather that they be solved on the major and minor axes of the jet cross section at each streamwise position. In this way the required computations are greatly reduced. Although the present analysis is laminar, references are made to turbulent experiments to point out qualitative agreement between the two.

Content

A schematic drawing of the flowfield geometry is presented in Fig. 1. The jet issues from an elliptical nozzle into a coflowing stream of velocity u_e. The Navier-Stokes and continuity equations are written in terms of an elliptical coordinate system, ξ, η, ζ. It is assumed that the isovels (i.e. lines of constant streamwise velocity) are elliptical, with arbitrary aspect ratio, and coincide with the coordinate surfaces ζ = constant. The assumption of elliptical isovels can be simply justified since Trentacoste and Sforza[1] have shown that even the isovels downstream of a turbulent rectangular jet become elliptical at some distance from the orifice. In addition, at the jet orifice the isovels are clearly elliptical since the shape of the orifice is elliptical and far downstream the isovels are circular. The natural transition between these limits is an elliptical isovel structure.

Thus the desired coordinate system has the property that the surfaces ζ = constant are cylinders of elliptical cross section extending in the streamwise direction. The surfaces ξ = constant correspond exactly to x = constant, i.e., to planes normal to the streamwise direction. The surfaces η = constant must be found from the definition of ζ such that the two families of surfaces are always orthogonal. The definition of ζ is

$$\zeta = [y^2 + \alpha^2 z^2]^{1/2}$$

where α is the aspect ratio of the local ellipse. To make the calculation of the local value of the transverse coordinate η tractable, derivatives of α are neglected. The slope of the ellipse is $(dz/dy)_{\zeta = \text{constant}}$. Its negative reciprocal is $(dz/dy)_{\eta = \text{constant}}$ and is integrated to obtain

$$\eta = z/y^{\alpha^2}$$

In addition, the assumption that the ellipses are isovels implies that

$$\partial u/\partial \eta = 0$$

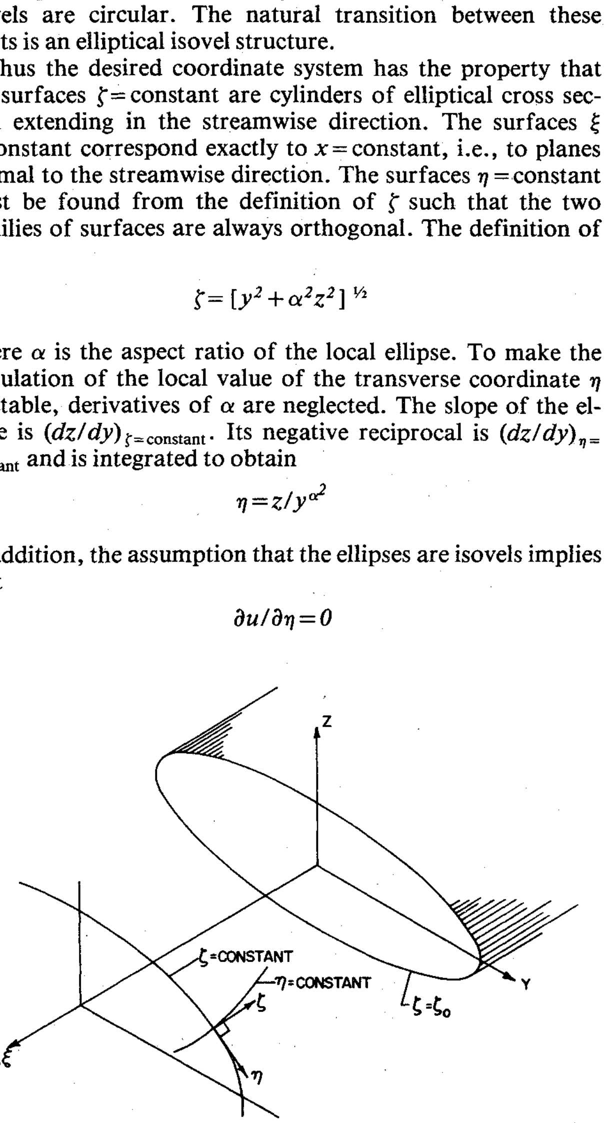

Fig. 1 Elliptical nozzle exit and elliptical coordinate system.

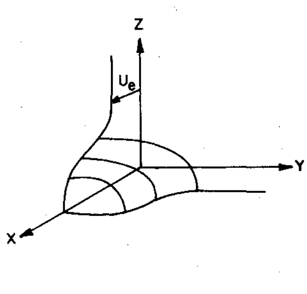

Fig. 2 Velocity profiles in two planes showing elliptical isovels.

Received October 15, 1974; synoptic received August 28, 1975. Full paper available from National Technical Information Service, Springfield, Va. 22151 as N75-31393 at the standard price (available upon request). Special thanks are due to B.P. Quinn and D.A. Lee for many enlightening conversations. The contributions of P.M. Bevilaqua, and C.N. Eastlake II for several discussions are also appreciated.

Index categories: Jets, Wakes, and Viscid-Inviscid Flow Interactions.

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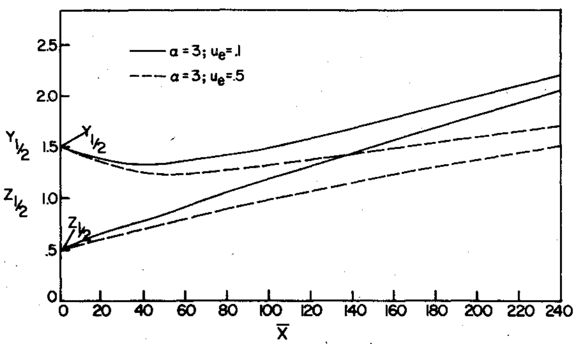

Fig. 3 Effect of coflowing stream on half width growth.

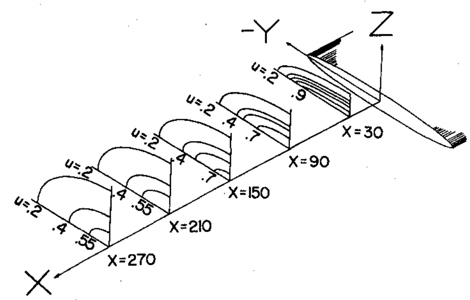

Fig. 4 Isometric of the isovel structure of an aspect ratio 10 jet.

By the use of this condition, the assumption on α and an order of magnitude analysis, the equations are reduced to two equations and two unknowns, u, v. The equations resemble those for a two-dimensional (ξ,ζ) boundary-layer flow problem with several additional terms. All references to the transverse coordinate η are lost in the assumptions reducing the complexity of the equations. This fact makes it possible to greatly reduce the amount of calculation required to solve the system of equations. Instead of solving the equations in the full three-dimensional space, it is only necessary to solve them in the xy and xz planes as shown in Fig. 2. Knowing the solution in these planes means the solution in known in the whole three-dimensional space, since the velocity profiles in the 2 planes are connected by isovels of elliptical shape. The aspect ratios of the elliptical isovels so obtained vary throughout the jet cross section (as well as downstream) and are completely determined by a comparison of the two velocity profiles on the axes.

The variation of the half widths in the streamwise direction is shown in Fig. 3 for two different values of the coflowing stream. The half widths on the major axis contract to some minimum position before they grow while those on the minor axis grow monotonically. Far downstream the half widths approach each other and the axisymmetric far field solution. It may be noted that the higher the value of the coflowing stream, the smaller are the half widths and the lower is their rate of growth. Essentially this means that the momentum in the jets does not spread as rapidly into faster coflowing streams. Another point of interest is that, for different levels of the coflowing stream, the half widths approach different asymptotic growth rates far downstream. This is related to the fact that the half width growth rate of any jet varies with the magnitude of the coflowing stream. As the velocity of the coflowing stream increases, the rate of half width growth decreases due to the lower shear about the periphery of the jet.

For higher aspect ratios the contraction is more pronounced and occurs farther downstream. The growth of the half width on the minor axis is affected only slightly, being initially greater but slowing until it is finally approximately the same as the lower aspect ratio case. The asymptotic limit of the half width growth far downstream is (as is the case with all the variables) the axisymmetric jet value. That is, far downstream the two half widths will grow at the same rate and have the same value.

It should be noted that the decrease in half width of the far side of the jet is an important result since it is part of the price which must be paid for rapid mixing of the jet. That is, the faster the half width growth of the near side of the jet, the greater is the half width decrease of the far side of the jet. This may be clearly seen in the turbulent jet results of Eastlake,[3] where the jet exit shape was modified to increase the mixing. The increased mixing rate caused the near side half width to grow more rapidly while the far side half width decrease was substantially exaggerated.

The importance of the half width shrink phenomenon is demonstrated by the thrust augmenting ejector experiments performed by Quinn.[4] The high aspect ratio primary nozzles originally employed could not penetrate to the center of the ejector because of the shrink effect. Extending the nozzles to the ejector centerline avoided this effect and resulted in a substantial increase in performance.

An overall view of the velocity field of a three-dimensional laminar jet is shown in Fig. 4, an isometric view of the isovel pattern due to a jet of aspect ratio 10 exhausting into a coflowing stream of 1/10 the jet velocity. The distance along the x axis has been scaled down so as to make the figure tractable. It can be clearly seen that the jet grows more rapidly along its minor axis than its major axis, and that the jet progresses steadily toward the axisymmetric flow condition.

References

[1] Trentacoste, N. and Sforza, P.M., "Further Experimental Results for Three-Dimensional Free Jets," *AIAA Journal*, Vol. 5, May 1967, pp. 885-891.

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[3] Eastlake, C.N., II, "The Macroscopic Characteristics of Some Subsonic Nozzles and the Three-Dimensional Turbulent Jets They Produce," Aerospace Research Lab., Rept. 71-0058, March 1971, Wright-Patterson Air Force Base, Ohio.

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